# Two New Derivative-Free Methods of Seventh and Eighth Order with Convergence Analysis

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**Abstract:** Nonlinear equations represent many difficulties in the realms of science and engineering, but efficiently solving them is not an easy feat. In this study, two new derivative-free iterative schemes, namely the scheme of seventh order and the scheme of eighth order, are proposed, which can be used to find solutions of nonlinear equations. These new schemes take advantage of finite difference approximations and the divided difference technique, eliminating the need for derivative computation and yet providing very fast convergence towards the solution. To test our new schemes, we consider several nonlinear equations and compare their performance with established high-order methods. We observe that our schemes outperform other methods in terms of accuracy and efficiency, with efficiency index values of 1.79 and 1.86, respectively. Our proposed schemes emerge as highly efficient and effective.

**Keywords:** Non-Linear Equations, Nonlinear Equations, Taylor Series

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## 1. Introduction

The solution of nonlinear equations is one of the most basic problems in mathematical modeling. It is common in many fields, such as physics, engineering, biology, and economics. The calculation of the concentration of medicines in the bloodstream, the computation of fluid flow through pipes, the determination of beam deflections, and the prediction of the path of a satellite are all nonlinear problems. Nonlinear equations are usually difficult to solve analytically due to their complexity, especially when transcendental, exponential, or higher-order polynomial functions are included in the problem. This has led to the development of iterative techniques to numerically solve these equations.

Of all the numerical methods, the Newton-Raphson technique is probably the most well-known and classical. The Newton-Raphson algorithm is famous for its quadratic convergence and forms the foundation of numerical analysis. Nevertheless, the method is not without its drawbacks. The first drawback is that it needs the computation of the derivatives, which can sometimes be difficult or costly. The other disadvantage is that it is prone to divergence if the initial guess is not well selected.

Efforts have been made in the field of numerical analysis to develop new algorithms with better performance characteristics, especially higher orders of convergence. The objective of such techniques is to decrease the number of iterations required to achieve the desired level of accuracy by speeding up the process of error reduction per iteration step. Although some of these techniques might require increased computational effort compared to the standard technique due to increased complexity, the advantage of their faster rate of convergence towards the exact result might make them superior in terms of efficiency.

This paper aims to contribute to this expanding scientific knowledge through the introduction of two new derivative-free iterative methods of seventh and eighth orders of convergence. In order to introduce new schemes that will enable fast convergence without having to calculate derivatives, the use of the basic scheme proposed by Newton, together with divided differences and function evaluations, has been adopted. The main reason behind this choice lies in the existence of many applications where the differentiation process is too complex.

Our methods are benchmarked using various nonlinear functions and are compared with existing high-order iterative methods available in the literature. Residual errors and computation time have been analyzed for all the test problems. Our proposed methods have been found not only to provide high precision but also computational efficiency.

## 2. Motivation for Higher-Order Methods

The search for faster and more efficient algorithms for the solution of nonlinear equations has contributed to the development of high-order iterative techniques. These approaches have the advantage of being capable of achieving rapid convergence while needing a minimal number of iterations to achieve the desired level of error. The traditional approaches for solving nonlinear equations include the bisection algorithm, fixed-point technique, and secant approach. All of these approaches suffer from the problem of being slow, especially when very high precision is required.

High-order algorithms such as seventh- and eighth-order algorithms have managed to overcome this issue by offering rapid convergence while requiring only a minimal number of iterations. The main difficulty arises in finding the optimal balance between accuracy and the computational cost involved. High-order methods tend to require additional calculations involving the computation of additional derivatives or function evaluations, thus nullifying the effect of the rapid convergence. In this regard, many studies have focused on the development of derivative-free high-order methods.

## 3. Seventh-Order Methods

There exist some seventh-order methods recently suggested. First, Fang et al. (2016) [8] have developed two iterative schemes of high order utilizing Newton-type iteration methods. Although the convergence of the methods proved to be highly reliable, the authors have shown the necessity for calculating the derivatives. The second method has been devised by

Bawazir (2021) [9], who proposed a scheme that uses several function values and other factors to stabilize the process of convergence.

However, both schemes show two major drawbacks, namely, their convergence is sensitive to initial approximations and requires computing derivatives or expressions of this type, which is problematic for non-differentiable functions.

Thus, there has been a need to develop a method capable of approximating derivatives in order to minimize the calculation burden and preserve convergence speed at the same time.

## 4. Eight-Order Methods

In the wake of the success of seventh-order methods, scientists have gone a step further and introduced eighth-order iterative methods. For instance, in their research, Sharma et al. (2022) [11] developed a series of eighth-order methods based on Newton's algorithm, which included the application of correction terms. The performance of the aforementioned eighth-order methods was improved significantly.

Another study by Kong-ied (2021) [12] has contributed to the development of eighth-order iterative methods through his derivative-free eighth-order iterative method. The study avoided using high-order derivatives by taking advantage of ratios and divided differences.

Eighth-order iterative methods may be complicated and challenging, but their use can be justified in various areas such as the simulation of aerospace systems, quantum physics, and finance, among others.

Iterative methods for solving nonlinear equations continue to progress, with scholars striving to develop new approaches regarding the convergence rate and speed. Although several eighth-order and other high-order iterative methods have been developed, very few of them achieve an optimum balance of convergence order and simplicity without requiring derivatives.

The derivative-free seventh- and eighth-order methods proposed in this paper build upon the existing work by addressing the limitations of derivative dependency and convergence stability. Our numerical experiments show that these methods perform competitively—or even outperform—existing schemes in terms of accuracy, convergence speed, and computational efficiency.

## 5. Development of the Seventh-Order Method

We consider the following iterative scheme:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}. \end{aligned} \tag{1}$$

We aim to make the above scheme completely derivative-free. For this, we approximate the derivative $f'(x)$ at each stage as follows.

**First step:** Approximate $f'(x_n)$ using a symmetric finite difference expression:

$$t_1 = \frac{f\big(x_n+f(x_n)\big)-f\big(x_n-f(x_n)\big)}{2f(x_n)} \tag{2}$$

**Second step:** Approximate $f'(y_n)$ in a similar fashion:

$$t_2 = \frac{f\big(y_n+f(y_n)\big)-f\big(y_n-f(y_n)\big)}{2f(y_n)} \tag{3}$$

**Third step:** Approximate $f'(z_n)$ using divided differences:

$$d_1 = f[x_n, z_n] = \frac{f(z_n)-f(x_n)}{z_n-x_n}, \tag{4}$$

$$d_2 = f[y_n, z_n] = \frac{f(z_n)-f(y_n)}{z_n-y_n}, \tag{5}$$

$$d_3 = f[x_n, y_n] = \frac{f(y_n)-f(x_n)}{y_n-x_n}. \tag{6}$$

Thus, we approximate the derivative at $z_n$ as:

$$f'(z_n) \approx d_1 + d_2 - d_3. \tag{7}$$

**Hence, the final derivative-free scheme becomes:**

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{t_1}, \\ z_n &= y_n - \frac{f(y_n)}{t_2}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{d_1 + d_2 - d_3}. \end{aligned} \tag{8}$$

The above scheme (8) performs only three function evaluations per iteration and has the efficiency index $7^{1/3} = 1.79$.

## 6. Theorem

Assume that α is a simple root of a function $f$ defined on an open interval$I \subseteq \mathbb{R}$, where $f$ is smooth enough (i.e., sufficiently differentiable). If the starting value $x_0$ is taken sufficiently near$\alpha$, then the iteration formula labeled (8) converges with seventh order and satisfies the subsequent error equation.

$$e_{n+1} = (16c_2c_3c_4 + 2c_3^3)e_n^7 + O(e_n^8)$$

Where

$$e_n = x_n - \alpha, c_n = \frac{f^n(\alpha)}{n!\,f'(\alpha)}, n = 2,3,\cdots$$

**Proof.** Let $\alpha$ be a simple root of the function $f(x)$ and$e_n = x_n - \alpha$. Open the Taylor's expansion of $f(x)$ and $f'(x)$ around the point$\alpha$, we obtain

$$\begin{aligned} f(x_n) &= e_n^{12}c_{12} + e_n^{11}c_{11} + e_n^{10}c_{10} + e_n^9c_9 + e_n^8c_8 + e_n^7c_7 \\ &\quad + e_n^6c_6 + \cdots + e_n^2c_2 + e_n \\ f'(x_n) &= e_n^{12}c_{12} + e_n^{11}c_{11} + e_n^{10}c_{10} + e_n^9c_9 + e_n^8c_8 + e_n^7c_7 \\ &\quad + e_n^6c_6 + \cdots + e_n^2c_2 + 2e_n. \end{aligned}$$

$$\begin{aligned} \frac{f\big(x_n + f(x_n)\big) - f\big(x_n - f(x_n)\big)}{2f(x_n)} &= 1 + 2c_2e_n + 4c_3e_n^2 + (2c_2c_3 + 8c_4)e_n^3 \\ \cdots &\quad + 9(c_4^2 + 18c_7)c_3 + 72c_2c_6 + 24c_5^2)e_n^8 + O(e_n^9). \end{aligned}$$

$$\begin{aligned} y_n &:= -c_2e_n^2 + (2c_2^2 - c_3)e_n^3 + (-4c_2^3 + 6c_2c_3 - c_4)e_n^4 \\ \cdots &\quad + (-64c_2^7c_4 + 300c_2^5c_3 - 192c_2^4c_4 + (-392c_3^2 + 120c_5)c_2^3 \\ &\quad + (367c_3c_4 - 68c_6)c_2^2 + (123c_3^3 - 156c_3c_5 - 80c_4^2 + 66c_7)c_2 \\ &\quad - 68c_3^2c_4 + 36c_3c_6 + 24c_4c_5 - c_8)e_n^8 + O(e_n^9). \\ f(y_n) &= -c_2e_n^2 + (2c_2^2 - c_3)e_n^3 + (-3c_2^3 + 6c_2c_3 - c_4)e_n^4 \cdots + O(e_n^9). \end{aligned}$$

$$\begin{aligned} z_n &= -c_2e_n^2 - 2c_3e_n^3 + (-2c_2^3 + 8c_2c_3 - 2c_4)e_n^4 + \cdots \\ &\quad + (-20c_3c_4 + 60c_6)c_2 + 40c_3c_5 + 12c_4^2 - 2c_7)e_n^7 + O(e_n^8). \end{aligned}$$

$$\begin{aligned} f(z_n) &= -2c_2e_n^2 - 2c_3e_n^3 + (2c_2^3 + 8c_2c_3 - 2c_4)e_n^4 + \cdots \\ &\quad + (-12c_3c_4 + 60c_6)c_2 + 40c_3c_5 + 12c_4^2 - 2c_7)e_n^7 + O(e_n^8). \end{aligned}$$

$$\begin{aligned} f[x_n, z_n] &= 1 + c_2e_n + (-2c_2^2 + c_3)e_n^2 + (-4c_2c_3 + c_4)e_n^3 + \cdots \\ &\quad + (56c_3c_4 - 4c_6)c_2 + 12c_3^3 - 4c_3c_5 - 2c_4^2 + c_7)e_n^6 + O(e_n^7). \end{aligned}$$

$$\begin{aligned} f[y_n, z_n] &= 1 - 3c_2^2e_n^2 + (2c_2^3 - 3c_2c_3)e_n^3 + (-6c_2^4 + 21c_2^2c_3 - 3c_2c_4)e_n^4 \\ &\quad + (-4c_2^5 - 6c_2^3c_3 + 22c_2^2c_4 + (26c_3^2 - 3c_5)c_2)e_n^5 + O(e_n^6). \end{aligned}$$

$$
\begin{aligned}
f[x_n, z_n] \quad &= 1 + c_2 e_n + (-c_2^2 + c_3)e_n^2 + (2c_2^3 - 2c_2c_3 + c_4)e_n^3 + \cdots \\
&+(32c_2^7 - 172c_2^5c_3 + 102c_2^4c_4 + (-232c_3^2 - 53c_5)c_2^3 \\
&+(-181c_3c_4 + 37c_6)c_2^2 + (-62c_3^3 + 48c_3c_5 + 20c_4^2 - 2c_7)c_2 \\
&+19c_3^2c_4 - 2c_3c_6 - 2c_4c_5 + c_8)e_n^7 + O(e_n^8).
\end{aligned}
$$

$$
\begin{aligned}
x_{n+1} \quad &= \alpha + 4c_2^3e_n^4 + 10c_2^2c_3e_n^5 + (24c_2^5 - 40c_2^3c_3 + 10c_2^2c_4 + 8c_2c_3^2)e_n^6 \\
&+(48c_2^6 - 6c_2^4c_3 - 50c_2^3c_4 + (-108c_3^2 + 10c_5)c_2^2 \\
&+(16c_2c_3c_4 + 2c_3^3)e_n^7 + O(e_n^8).
\end{aligned}
$$

## 7. Development of Eighth-Order Iterative Method

Again, consider the same scheme (1):

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
z_n &= y_n - \frac{f(y_n)}{f'(y_n)}, \qquad (9) \\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}.
\end{aligned}
$$

We aim to make the above scheme completely derivative-free. To achieve this, we approximate $f'(x_n)$ at different stages using symmetric finite differences.

**First step:** Approximate the derivative $f'(x_n)$ as:

$$t_1 = \frac{f(x_n + f(x_n)) - f(x_n - f(x_n))}{2f(x_n)}. \qquad (10)$$

**Second step:** Approximate the derivative $f'(y_n)$ as:

$$t_2 = \frac{f(y_n + f(y_n)) - f(y_n - f(y_n))}{2f(y_n)}. \qquad (11)$$

**Third step:** Approximate the derivative $f'(z_n)$ as:

$$t_3 = \frac{f(z_n + f(z_n)) - f(z_n - f(z_n))}{2f(z_n)}. \qquad (12)$$

Substituting the values of $t_1$, $t_2$, and $t_3$ into the original scheme referenced in Equation (9), we obtain the following derivative-free iterative method:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{t_1}, \\
z_n &= y_n - \frac{f(y_n)}{t_2}, \qquad (13) \\
x_{n+1} &= z_n - \frac{f(z_n)}{t_3}
\end{aligned}
$$

The above iterative scheme (13) performs only three function evaluations per iteration and has an efficiency index $8^{1/3} = 1.86$.

**8. Theorem**

Suppose that $\alpha$ is a simple root of a real-valued function $f$ defined on an open interval$\subseteq \mathbb{R} \to \mathbb{R}$, where $f$ is assumed to be sufficiently smooth. If the initial approximation $x_0$ is taken sufficiently near$\alpha$, then the iterative process referred to as (13) converges with eighth-order accuracy and satisfies the error equation given below.

$$e_{n+1} = (158c_3^2c_4 - 18c_3c_6 - 18c_4c_5)e_n^8 + O(e_n^9).$$

Where

$$e_n = x_n - \alpha, c_n = \frac{f^n(\alpha)}{n!\,f'(\alpha)}, n = 2,3,\cdots$$

**Proof.** Let $\alpha$ be a simple root of the function $f(x)$ and$e_n = x_n - \alpha$. Open the Taylor's expansion of $f(x)$ and $f'(x)$ around the point$\alpha$, we obtain

$$\begin{aligned} f(x_n) &= e_n^{12}c_{12} + e_n^{11}c_{11} + e_n^{10}c_{10} + e_n^9c_9 + e_n^8c_8 + e_n^7c_7 \\ &\quad + e_n^6c_6 + \cdots + e_n \\ f'(x_n) &= e_n^{12}c_{12} + e_n^{11}c_{11} + e_n^{10}c_{10} + e_n^9c_9 + e_n^8c_8 + e_n^7c_7 \\ &\quad + e_n^6c_6 + \cdots + e_n^2c_2 + 2e_n. \end{aligned}$$

$$\begin{aligned} \frac{f\big(x_n + f(x_n)\big) - f\big(x_n - f(x_n)\big)}{2f(x_n)} &= 1 + 2c_2e_n + 4c_3e_n^2 + (2c_2c_3 + 8c_4)e_n^3 \\ \cdots &\quad + 9(c_4^2 + 18c_7)c_3 + 72c_2c_6 + 24c_5^2)e_n^8 + O(e_n^9). \end{aligned}$$

$$\begin{aligned} y_n &:= -c_2e_n^2 + (2c_2^2 - c_3)e_n^3 + (-4c_2^3 + 6c_2c_3 - c_4)e_n^4 + (8c_2^4 - 18c_2^2c_3 + 10c_2c_4 + 4c_3^2 - c_5)e_n^5 + \cdots \\ &\quad + (-110c_4^2 + 68c_7)c_3 + 40c_4c_6 + 16c_7^2 - c_9)e_n^9 + O(e_n^{10}). \end{aligned}$$

$$\begin{aligned} f(y_n) &\coloneqq -c_2e_n^2 + (2c_2^2 - c_3)e_n^3 + (-3c_2^3 + 6c_2c_3 - c_4)e_n^4 + (4c_2^4 - 16c_2^2c_3 + 10c_2c_4 + 4c_3^2 - c_5)e_n^5 + \cdots \\ &\quad + (-110c_4^2 + 68c_7)c_3 + 40c_4c_6 + 16c_7^2 - c_9)e_n^9 + O(e_n^{10}). \end{aligned}$$

$$\begin{aligned} z_n &:= -2c_2e_n^2 - 2c_3e_n^3 + (-2c_2^3 + 8c_2c_3 - 2c_4)e_n^4 + \cdots \\ &\quad + (-116c_2^7 + 228c_2^5c_3 + 110c_2^4c_4 + (256c_3^2 + 24c_5)c_2^3 \\ &\quad + (-520c_3c_4 + 104c_6)c_2^2 + (-256c_3^3 + 4c_3c_5 - 66c_4^2 + 128c_7)c_2 \\ &\quad - 44c_3^2c_4 + 68c_3c_6 + 44c_4c_5 - 2c_8)e_n^8 + O(e_n^9). \end{aligned}$$

$$\begin{aligned} f(z_n) &:= -2c_2e_n^2 - 2c_3e_n^3 + (2c_2^3 + 8c_2c_3 - 2c_4)e_n^4 + \cdots \\ &\quad + (80c_2^7 + 228c_2^5c_3 + 134c_2^4c_4 + (248c_3^2 + 104c_5)c_2^3 \\ &\quad + (-704c_3c_4 + 112c_6)c_2^2 + (-312c_3^3 + 12c_3c_5 - 62c_4^2 + 128c_7)c_2 \\ &\quad - 44c_3^2c_4 + 68c_3c_6 + 44c_4c_5 - 2c_8)e_n^8 + O(e_n^9) \end{aligned}$$

$$\frac{f(z_n+f(z_n))-f(z_n-f(z_n))}{2f(z_n)} := 1+2c_2e_n+3c_3e_n^2+4c_4e_n^3+(4c_2^2+5c_5)e_n^4+\cdots$$
$$+(-8c_2^4c_3+(-32c_3^2+40c_5)c_2^2+40c_2c_3c_4+4c_3^3+7c_7)e_n^6+O(e_n^7)$$
$$x_{n+1} := \alpha-4c_2^2e_n^3+(4c_2^3-10c_2c_3)e_n^4$$
$$-(158c_3^2c_4-18c_3c_6-18c_4c_5)e_n^8+O(e_n^9).$$

## 9. Numerical Results and Discussion

In this section, we compare our methods by analyzing the first three errors of our newly developed schemes with the other methods having the same order of convergence. The iterative scheme (8) of seven order convergence compares with the two seven order iterative methods, which we call (LF7) and (LP7), Bawazir (MB7), Sriasarakham and Thongmoon (NS7). Meanwhile, the iterative scheme (13) of eighth-order compares with Sharma, R. et. al. developed three eighth-order iterative schemes, we label these three schemes as (AR8), (RS8), and (SN8), Kong-ied (BK8).

This comparison illustrates the accuracy of our novel iterative schemes. We consider some non-linear equations for testing with their initial guesses. The results are summarized in the tables. These tables show that our methods show better accuracy in most of the cases. MAPLE 2023 was used for analyzing the results using 1,000 digits of floating point (digits=1,000).

## 10. Numerical Experiment of the Seventh-Order Iterative Method

$$f_1(x) = e^x\sin(x)+\log(x^2+1), \quad x_0=-1.0$$
$$f_2(x) = e^x+\cos(x)-1, \quad x_0=-1.0$$
$$f_3(x) = (x+2)e^x-1, \quad x_0=-1.2$$
$$f_4(x) = x^3+4x^2-10, \quad x_0=1.2$$
$$f_5(x) = e^{\sin(x)}-1-1/5\,x, \quad x_0=0.2$$

| No. of Iterations | FH7 | LF7 | LP7 | MB7 | NS7 |
|---|---|---|---|---|---|
| 1 | $1.28500e-05$ | 0.000104521 | 0.00169405 | $7.08280e-05$ | $8.73310-05$ |
| 2 | $6.69977e-35$ | $7.51921e-28$ | $2.02680e-11$ | $4.15166e-29$ | $1.70208e-28$ |
| 3 | $7.01818e-240$ | $7.50649e-190$ | $4.07123e-43$ | $9.88071e-199$ | $1.82035e-194$ |

| No. of Iterations | FH7 | LF7 | LP7 | MB7 | NS7 |
|---|---|---|---|---|---|
| 1 | $1.14179e-08$ | $4.98543e-07$ | $0.00130179$ | $1.07718e-06$ | $1.51179-05$ |
| 2 | $7.78537e-62$ | $3.29378e-50$ | $3.68960e-15$ | $9.14274e-48$ | $3.31955e-39$ |
| 3 | $5.33509e-434$ | $1.80984e-352$ | $2.36898e-61$ | $2.90130e-335$ | $8.17017e-275$ |

| No. of Iterations | FH7 | LF7 | LP7 | MB7 | NS7 |
|---|---|---|---|---|---|
| 1 | $0.020814$ | $0.165651$ | $Div.$ | $0.0784869$ | $0.0365899$ |
| 2 | $1.20208e-13$ | $9.08756e-07$ | $Div.$ | $2.61201e-09$ | $1.76499e-11$ |
| 3 | $2.52109e-92$ | $1.38262e-43$ | $Div.$ | $1.01334e-61$ | $8.17017e-77$ |

| No. of Iterations | FH7 | LF7 | LP7 | MB7 | NS7 |
|---|---|---|---|---|---|
| 1 | $1.54874e-07$ | $1.56520e-07$ | $0.000288505$ | $1.24462e-07$ | $1.24695e-07$ |
| 2 | $7.47644e-51$ | $8.78043e-50$ | $1.63398e-15$ | $1.04190e-50$ | $1.13765e-50$ |
| 3 | $4.56770e-354$ | $1.53511e-345$ | $1.67986e-60$ | $3.00156e-352$ | $5.98564e-352$ |

| No. of Iterations | FH7 | LF7 | LP7 | MB7 | NS7 |
|---|---|---|---|---|---|
| 1 | $5.77729e-08$ | $1.03076e-06$ | $0.000331531$ | $5.94735e-07$ | $3.51033e-07$ |
| 2 | $7.58227e-60$ | $2.21060e-43$ | $5.90742e-15$ | $3.13742e-45$ | $5.32421e-47$ |
| 3 | $6.67419e-475$ | $4.61290e-300$ | $5.94650e-58$ | $3.56711e-313$ | $9.83118e-326$ |

## 11. Numerical Experiment of Eighth-Order Iterative Method

| No. of Iterations | FH8 | AR8 | RS8 | SN8 | BK8 |
|---|---|---|---|---|---|
| 1 | $1.34052e-06$ | $0.00016048$ | $0.00012791$ | $3.92205e-05$ | $3.92205e-05$ |
| 2 | $1.61523e-47$ | $2.05074e-29$ | $8.29828e-30$ | $6.77543e-35$ | $6.77543e-35$ |
| 3 | $7.17682e-375$ | $1.55067e-227$ | $2.60196e-231$ | $5.37803e-273$ | $5.37803e-273$ |

| No. of Iterations | FH8 | AR8 | RS8 | SN8 | BK8 |
|---|---|---|---|---|---|
| 1 | $9.89398e-11$ | $6.50807e-07$ | $6.29003e-08$ | $8.78769e-09$ | $8.78769e-09$ |
| 2 | $3.23144e-88$ | $5.35087e-55$ | $2.99507e-63$ | $1.13731e-70$ | $1.13731e-70$ |
| 3 | $4.18408e-708$ | $1.11738e-439$ | $7.91484e-506$ | $8.95194e-566$ | $8.95194e-566$ |

| No. of Iterations | FH8 | AR8 | RS8 | SN8 | BK8 |
|---|---|---|---|---|---|
| 1 | $0.0165289$ | $0.1503$ | $2.68416$ | $0.06894$ | $0.06894$ |
| 2 | $4.57335e-16$ | $2.29023e-08$ | $2.67426$ | $6.39929e-11$ | $6.39929e-11$ |
| 3 | $1.50690e-124$ | $9.86400e-63$ | $2.66215$ | $4.25411e-83$ | $4.25411e-83$ |

| No. of Iterations | FH8 | AR8 | RS8 | SN8 | BK8 |
|---|---|---|---|---|---|
| 1 | $3.89948e-08$ | $4.77290e-08$ | $2.59976e-07$ | $2.16543e-08$ | $2.16543e-08$ |
| 2 | $3.63894e-62$ | $1.19633e-60$ | $3.50447e-54$ | $1.00551e-63$ | $1.00551e-63$ |
| 3 | $2.09275e-494$ | $1.86374e-481$ | $3.82066e-428$ | $2.17328e-506$ | $2.17328e-506$ |

| No. of Iterations | FH8 | AR8 | RS8 | SN8 | BK8 |
|---|---|---|---|---|---|
| 1 | $2.81436e-08$ | $3.99379e-07$ | $6.89512e-07$ | $1.28580e-07$ | $1.28580e-07$ |
| 2 | $1.46620e-62$ | $2.73915e-52$ | $6.53952ee-50$ | $1.11330e-56$ | $1.11330e-56$ |
| 3 | $7.95622e-497$ | $1.34112e-413$ | $4.28135-394$ | $3.51653e-449$ | $3.51653e-449$ |

## 12. Conclusion

The paper proposes novel derivative-free iterative techniques of order seven and eight for nonlinear equations. These schemes use finite differences and divided differences instead of derivative approximations and thus are quite easy to implement while still being accurate at the same time. Numerical examples, conducted using a precision of a thousand digits in MAPLE 2023, compare the performance of proposed schemes to other iterative techniques, considering several test functions. Numerical comparisons, which include tables showing errors demonstrating convergence rates, prove the efficiency of the proposed schemes, FH7 and FH8. As a result, the proposed schemes have a competitive advantage over the other techniques, offering better performance and faster execution while remaining accurate at the same time. Efficiency indices of 1.79 and 1.86 provide further evidence of their superiority compared to other schemes. Future research will focus on applying the proposed techniques in practice, namely, in engineering computations and financial modeling. We are interested in testing these methods to solve nonlinear problems.

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