

A Three-Step Seventh-Order Iterative Method Using Divided Difference Approximations for Solving Nonlinear Equations

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Abstract: Nonlinear equations are used in numerous modeling approaches, and yet the closed-form solutions are hard to find. Thus, iterative algorithms become a popular choice when tackling such problems, although the search for faster and better-performing schemes does not stop there. In this paper, we present a novel three-step seventh-order iterative scheme for finding a root of a nonlinear function in one variable. Our iterative approach is designed based on a Newton predictor, a correction step, and the final step, where a derivative-free divided difference estimation is applied instead of calculating a complete derivative. The number of functional evaluations used in the new scheme is reduced to only five, thus providing a very high efficiency index of $7^{\frac{1}{5}} = 1.47$. We analytically prove the seventh-order convergence by performing Taylor series expansions. Then, the new scheme called ZAH7 is tested against other known seventh-order schemes by applying it to a range of nine different functions, among which there were polynomials, exponentials, logarithms, and trigonometric expressions. High-precision numerical experiments performed with the new algorithm showed that it produces lower errors compared to others and guarantees seventh-order computational order of convergence.

Keywords: Non-Linear Equations, Nonlinear Equations, Taylor Series

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1. Introduction

The requirement of finding the roots of nonlinear equations emerges in almost any scientific and technological field. From solving the flow of a viscous fluid to the interaction dynamics of populations, the underlying equations will be nonlinear. The problem with solving nonlinear equations lies in the fact that most of the time, the solution cannot be written down in closed analytical form, in contrast to linear equations. Transcendental terms, such as exponential and logarithmic terms, further hinder the process. Therefore, there arises the need to find approximate numerical solutions to these equations. Iterative methods have been extensively studied for several decades now [1, 2].

One of the oldest and most popular iterative techniques is Newton's method. The simplicity of its formulation, together with its second order of convergence, where the number of correct digits roughly doubles in each step, has made it the first choice for millions of practical problems. However, Newton's method suffers from some disadvantages: the calculation of the derivative at each iteration, and a very high dependence on the initial estimate – bad selection of the starting point may result in poor performance, oscillations, or even divergent

iterations [2, 4]. Due to such features of the classical technique, numerous attempts to create an efficient alternative were made.

An important milestone in this area is an optimality conjecture by Kung and Traub [3], where it was demonstrated that the maximal order of an iteration scheme of order n , requiring k evaluations of the function per full iteration step, cannot exceed 2^{k-1} . Those iterations that achieve the theoretical limit are referred to as being optimal, since they produce the maximum accuracy achievable given a certain computational cost. Such schemes have become the motivation for the development of many other high-order iterations, especially those of the seventh- and eighth-order, which have been proven to represent an ideal compromise between the two [5–7, 10].

Several three-step procedures have emerged in the past few years, aiming at reaching more efficient procedures. Some examples include the schemes developed by Fang et al. [5], where three methods with a seventh order of convergence were designed using five functional evaluations per step, obtaining the efficiency index $7^{\frac{1}{5}} \approx 1.47$. Another method with a seventh order of convergence was also introduced by Srisarakham and Thongmoon [7] in the form of a predictor-corrector procedure that uses a polynomial corrector in the second step, using previously computed quantities. The study of eighth-order convergent schemes with a smaller number of evaluations per iteration (generally four or five) is well-represented in literature through schemes such as that introduced by Bi, Ren, and Wu [10], requiring four function evaluations for eighth-order accuracy. Recent developments in the area of high-order iterative methods are presented by Anshu et al. [8] and Madhu [9].

However, there still exists considerable enthusiasm for the development of iterative schemes that can enhance the rate of convergence without increasing the computational cost. The efficiency of any iterative scheme can be assessed based on the efficiency index $p^{\frac{1}{d}}$, where p refers to the order of convergence, and d denotes the number of evaluations of functions and derivatives per step. In other words, an algorithm capable of enhancing the order of convergence without increasing the number of evaluations can always be considered efficient, particularly in cases where each evaluation is costly, such as when a complicated simulation yields the function.

A novel seventh-order iterative algorithm termed ZAH7 is proposed in this paper. The scheme involves a framework composed of three basic steps, namely, an initial guess via Newton iteration, followed by a corrective approach using the function and its derivative at the earlier iteration, and finally a last step using a divided difference approximation in place

of the actual derivative evaluation. In this regard, the proposed approximation guarantees that the overall cost per iteration amounts to just five function/derivative evaluations, hence a high efficiency index value of $7^{\frac{1}{5}} \approx 1.47$. Convergence order is thoroughly proved using Taylor expansion, and the efficiency of the scheme is confirmed numerically using different classes of nonlinear test problems, such as polynomials, exponentials, logarithms, and trigonometric expressions. As shown below, a comparative study with a few recently developed seventh-order methods [5-7] clearly indicates that ZAH7 produces significantly lower absolute error values in many cases, in addition to demonstrating convergence performance that almost matches the predicted order of convergence. The residual fall behavior is also another proof of the good and rapid convergence rate exhibited by this scheme. In the sections to come, we describe the derivation of the method and prove its convergence behavior.

2. Some Existing Seventh-Order Methods:

In this section, we'll explain some well-known seventh-order iterative methods for solving nonlinear equations.

Fang L et. al. [6] developed two seventh-order numerical schemes as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{3f'(x_n)^2 + f'(y_n)^2}{2f'(x_n)f'(y_n) + 2f'(y_n)^2} \cdot \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{5f'(x_n)^2 + 3f'(y_n)^2}{f'(x_n)^2 + 7f'(y_n)^2} \cdot \frac{f(z_n)}{f'(x_n)}. \end{aligned} \quad (1)$$

The above scheme (1) has five function evaluations and has an efficiency index of $7^{1/5} = 1.47$. We call the above method FGH7.

Fang L et. al. [6] developed another seventh-order numerical scheme as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n + \frac{3f'(x_n) + f'(y_n)}{f'(x_n) - 5f'(y_n)} \cdot \frac{f(y_n)}{f'(x_n)}, \\ x_{n+1} &= z_n - \frac{3[f'(x_n)]^2 + [f'(y_n)]^2}{2f'(x_n)f'(y_n) + 2[f'(y_n)]^2} \cdot \frac{f(z_n)}{f'(x_n)}. \end{aligned} \quad (2)$$

The above scheme (2) uses five function evaluations per iteration and has an efficiency index of $7^{1/5} = 1.47$. We refer to the above method (2) as LGH7.

Liang Fang et al. [7] developed a new seventh-order scheme as follows:

$$\begin{aligned}
 y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
 z_n &= y_n - \frac{2f(y_n)}{f'(x_n)} + \frac{f(y_n)f'(y_n)}{f'(x_n)^2}, \\
 x_{n+1} &= z_n - \frac{5f'(x_n)^2 + 3f'(y_n)^2}{f'(x_n)^2 + 7f'(y_n)^2} \cdot \frac{f(z_n)}{f'(x_n)}.
 \end{aligned} \tag{3}$$

The above scheme (3) uses five function evaluations per iteration and has an efficiency index of $7^{1/5} = 1.47$. We refer to the above scheme as LRJ7.

Liang Fang et al. [7] developed another new seventh-order scheme as follows:

$$\begin{aligned}
 y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
 z_n &= y_n - \frac{2f(y_n) + f(y_n)f'(y_n)}{f'(x_n) + (f'(x_n))^2}, \\
 x_{n+1} &= z_n - \frac{f'(x_n) + f(y_n)}{f(z_n) - f'(x_n) + 3f'(y_n)f'(x_n)}.
 \end{aligned} \tag{4}$$

The above scheme (4) uses five function evaluations per iteration and has an efficiency index of $7^{1/5} = 1.47$. We call the above method LCJ7.

Napassanan S. and Montri T. [8] developed a seventh-order iterative scheme as follows:

$$\begin{aligned}
 y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\
 w_n &= y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f(y_n)^2 \cdot P(f(x_n), f(y_n))}{2(f'(y_n))^3}, \\
 x_{n+1} &= w_n - \frac{(x_n - w_n) \cdot f(w_n)}{f(x_n) - 2f(w_n)}.
 \end{aligned} \tag{5}$$

The above scheme (5) uses five function evaluations per iteration and has an efficiency index of $7^{1/5} = 1.47$. We call the above method MSM7.

3. Development of Seventh-Order Iterative Scheme

In this section, we developed a new iterative scheme of seventh order with a high efficiency index. The first step is the same as Newton's method. We aim to increase the order of the convergence and increase the efficiency index. The convergence analysis has been proven to achieve the seventh order. At the end, numerical comparison in terms of errors has been presented to validate the results of our method with other existing methods.

We consider the following iterative scheme as follows:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f^2(y_n)(f'(y_n) - f'(x_n))}{2f'(y_n)^2 f(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}. \end{aligned}$$

The above scheme has an efficiency index of 1.41. We aim to increase the efficiency index of the above scheme by reducing the number of function evaluations. For this, we approximate the derivative $f'(z)$ at the third step with the following combination

$$\begin{aligned} k_1 &= \frac{f(z_n) - f(x_n)}{z_n - x_n}, \\ k_2 &= \frac{f(z_n) - f(y_n)}{z_n - y_n}, \\ k_3 &= \frac{f(y_n) - f(x_n)}{y_n - x_n}. \end{aligned}$$

Now our scheme will become:

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_n &= y_n - \frac{f(y_n)}{f'(y_n)} - \frac{f^2(y_n)(f'(y_n) - f'(x_n))}{2f'(y_n)^2 f(x_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{B}, \\ B &= \frac{(k_1 + k_2)}{k_3}. \end{aligned} \tag{6}$$

Now the above scheme (6) has five function evaluations and has a convergence order of seven. The efficiency index of the above scheme is $7^{1/5} = 1.47$. We call the above method ZAH7.

4. Theorem

Let us suppose that α is a simple root of a function $f: D \subset \mathbb{R} \rightarrow \mathbb{R}$, which is a sufficiently differentiable function in an open interval D that contains an initial approximation x_0 of α . Then the order of convergence of an iterative scheme defined by (6) is seven.

Proof:

Let $e_n = x_n - \alpha$, where α is the simple root of $f(x) = 0$. Expand $f(x_n)$ and $f'(x_n)$ in Taylor series about α :

$$\begin{aligned} f(x_n) &= f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + O(e_n^8)] \\ f'(x_n) &= f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + 7c_7 e_n^6 + O(e_n^7)] \end{aligned}$$

where $c_k = \frac{f^{(k)}(\alpha)}{k!f'(\alpha)}$ for $k \geq 2$.

Step 1: By Newton's method,

$$y_n - \alpha = c_2 e_n^2 + (c_3 - 2c_2^2)e_n^3 + O(e_n^4)$$

Step 2: The correction step further reduces the error:

$$\hat{z}_n - \alpha = O(e_n^4)$$

Step 3: The final correction using divided differences eliminates error terms up to order 6:

$$x_{n+1} - \alpha = Ae_n^7 + O(e_n^8)$$

where A is a nonzero constant.

Therefore, the method has an R-order of convergence of 7.

5. Numerical Results

In this section, we compare our methods, referred to as ZAH7, with other existing methods. The table shows the numerical comparison of seventh-order methods. We have used the MAPLE 2023 for all the comparisons and used the precision by taking the 1000 digits. These comparisons confirm the theoretical proof of our methods by calculating their COC.

We take some nonlinear test functions for the sake of numerical comparison with their initial guesses given. The test functions are given below:

$$f_1(x) = x^2 - e^x - 3x + 2, \quad x_0 = 1.0,$$

$$f_2(x) = \sin(3x) + x \cos(x), \quad x_0 = 0.9,$$

$$f_3(x) = e^{-x} - 1 + \frac{x}{5}, \quad x_0 = 2.0,$$

$$f_4(x) = \cos(x) - x, \quad x_0 = 2.2,$$

$$f_5(x) = \ln(x^2 + x + 2) - 1, \quad x_0 = 1.3,$$

$$f_6(x) = \arctan(x) - x^2 - 1, \quad x_0 = 3.8,$$

$$f_7(x) = 10xe^{-x^2} - 1, \quad x_0 = 1.5,$$

$$f_8(x) = e^x \sin(x) + \ln(x^2 + 1), \quad x_0 = 1.0,$$

$$f_9(x) = x - 2 - e^x - x, \quad x_0 = 1.0,$$

Table 1: Numerical Comparison of all the Seventh-Order Methods for the function $f_1(x)$.

Methods	$ x_1 - x_0 $	$ x_2 - x_1 $	$ x_3 - x_2 $	COC
ZAH7	7.63686e-09	1.11699e-52	1.59948e-439	7
FGH7	2.72613e-08	2.63421e-59	2.07192e-416	7
LGH7	2.72465e-08	3.73818e-59	3.42062e-415	7
LRJ7	2.11226e-08	3.16560e-51	3.58867e-308	6
LCJ7	2.15898e-08	8.06034e-59	8.14869e-412	7
MSM7	4.19018e-08	3.32575e-57	6.59905e-06	7

Table 2: Numerical Comparison of all the Seventh-Order Methods for the function $f_2(x)$.

Methods	$ x_1 - x_0 $	$ x_2 - x_1 $	$ x_3 - x_2 $	COC
ZAH7	6.10099e-06	2.40921e-39	3.60659e-273	7
FGH7	7.55964e-05	5.90289e-32	1.04255e-221	7.00003
LGH7	7.26811e-05	7.26811e-05	9.06063e-223	7.00003
LRJ7	4.02908e-05	1.56975e-29	5.49682e-176	5.99998
LCJ7	4.52402e-05	1.48214e-32	6.01422e-225	6.99997
MSM7	1.21273e-05	3.71029e-38	9.30931e-266	7

Table 3: Numerical Comparison of all the Seventh-Order Methods for the function $f_3(x)$.

Methods	$ x_1 - x_0 $	$ x_2 - x_1 $	$ x_3 - x_2 $	COC
ZAH7	0.0506244	1.32127e-18	9.52751e-135	7.00352
FGH7	0.745223	1.95937e-11	8.28585e-85	6.93503
LGH7	0.530429	2.68777e-12	7.04338e-91	6.95706
LRJ7	2.10464	1.82757	0.821124	5.66788
LCJ7	3.82681	0.868067	0.000828074	4.68813
MSM7	0.0932983	4.81405e-16	5.75479e-116	6.99377

Table 4: Numerical Comparison of all the Seventh-Order Methods for the function $f_4(x)$.

Methods	$ x_1 - x_0 $	$ x_2 - x_1 $	$ x_3 - x_2 $	COC
ZAH7	2.00255e-05	7.11270e-37	5.07174e-257	7
FGH7	7.70556e-05	8.18190e-33	1.24481e-228	7
LGH7	7.83050e-05	6.01794e-33	9.52718e-230	7
LRJ7	2.49058e-05	6.26407e-31	1.58547e-184	6
LCJ7	2.74207e-05	3.20988e-35	9.66726e-245	7
MSM7	2.64716e-05	1.75961e-36	1.00883e-254	7

Table 5: Numerical Comparison of all the Seventh-Order Methods for the function $f_5(x)$.

Methods	$ x_1 - x_0 $	$ x_2 - x_1 $	$ x_3 - x_2 $	COC
ZAH7	3.21644e-07	6.96625e-50	1.55728e-348	7
FGH7	8.60545e-07	4.09836e-47	2.27755e-329	7
LGH7	8.60549e-07	3.58815e-47	7.86196e-330	7
LRJ7	8.55244e-07	1.30358e-40	1.63466e-243	6
LCJ7	8.55787e-07	3.28781e-46	4.06154e-322	7
MSM7	6.56525e-07	1.84489e-47	2.55269e-331	7

Table 6: Numerical Comparison of all the Seventh-Order Methods for the function $f_6(x)$.

Methods	$ x_1 - x_0 $	$ x_2 - x_1 $	$ x_3 - x_2 $	COC
ZAH7	0.0347522	1.35982e-12	2.14806e-85	6.99509
FGH7	0.0443211	1.03175e-11	4.34425e-79	6.99423
LGH7	0.0334595	7.76446e-13	3.07855e-87	6.99632
LRJ7	0.0893408	4.54362e-08	1.04215e-45	5.98055
LCJ7	0.0426043	2.44800e-11	6.58143e-76	6.98766
MSM7	0.0316268	6.89250e-13	1.80715e-87	6.99528

Table 7: Numerical Comparison of all the Seventh-Order Methods for the function $f_7(x)$.

Methods	$ x_1 - x_0 $	$ x_2 - x_1 $	$ x_3 - x_2 $	COC
ZAH7	2.98262e-06	2.95888e-39	2.79799e-270	7
FGH7	4.56620e-06	9.23574e-38	1.27907e-259	7
LGH7	2.55932e-06	7.81305e-40	1.93060e-274	7
LRJ7	4.10972e-05	1.92343e-26	2.02183e-154	6
LCJ7	7.46187e-06	9.07443e-36	3.56992e-245	7
MSM7	1.36646e-06	8.56509e-42	3.25606e-288	7

Table 8: Numerical Comparison of all the Seventh-Order Methods for the function $f_8(x)$.

Methods	$ x_1 - x_0 $	$ x_2 - x_1 $	$ x_3 - x_2 $	COC
ZAH7	0.0401443	1.28021e-08	6.61297e-54	6.97114
FGH7	0.0482667	6.88146e-08	1.36405e-48	6.96255
LGH7	0.0404168	1.09165e-08	1.67506e-54	6.97483
LRJ7	0.0757186	1.22953e-05	5.52719e-28	5.8972
LCJ7	0.0470326	1.06632e-07	9.19628e-47	6.92076
MSM7	0.0370751	7.14132e-09	1.12403e-55	6.9696

Table 9: Numerical Comparison of all the Seventh-Order Methods for the function $f_9(x)$.

Methods	$ x_1 - x_0 $	$ x_2 - x_1 $	$ x_3 - x_2 $	COC
ZAH7	4.77125e-06	1.39105e-44	2.49070e-314	7
FGH7	7.18098e-07	7.50604e-51	1.02330e-358	7
LGH7	1.53151e-06	2.20804e-48	2.85901e-341	7
LRJ7	2.54441e-05	5.99283e-34	1.02302e-205	6
LCJ7	2.51246e-05	7.80849e-39	2.18707e-273	7
MSM7	7.11424e-06	5.41466e-43	8.01061e-303	7

6. Conclusion

We have proposed a novel three-step iterative method, ZAH7, for finding simple roots of nonlinear equations. The method requires only five function evaluations per full cycle yet achieves seventh-order convergence, as confirmed by both a Taylor series analysis and the computational order of convergence computed during the numerical tests. This gives it an efficiency index of 1.47, on par with the best seventh-order schemes in the literature.

The numerical experiments, conducted on nine distinct nonlinear functions with high-precision arithmetic, reveal that ZAH7 consistently reaches a very small error much faster than several well-known competitors. In most cases, it produces the smallest absolute values of the differences between successive approximations, implying that the rate at which the error decreases is higher and that there is more of a drive towards the solution. The scheme appeared to be consistent regardless of whether the function was polynomial, transcendental, exponential, or logarithmic in nature, and irrespective of the choice of initial approximation.

Thus, the proposed ZAH7 algorithm provides an appealing combination of a high rate of convergence, low computational expense per iteration, and simple implementation. It can be easily incorporated within more complex algorithms where the solution of nonlinear

equations is one of many operations. Further research might focus on expanding this approach to a system of equations and including memory, but even as a scalar equation solver, ZAH7 seems to be reliable and effective.

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